Becoming a Lumberjack - Using $\log_n m$

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♢1 Introduction

To become a lumberjack requires that one is a master of logs. Indeed, that is why you are here today. The logarithm is a very important operation. With its definition, problem solvers and mathematicians are able to deal with exponents in very nice ways. Namely, it becomes easy to remove the exponent of both sides of an equation, which could be a very helpful step to speed up your algebra and manipulate equalities and inequalities.

♢2 Theory

Definition (Logarithms). Let $n, k, m \in \mathbb{R}$ such that $n^k = m$ We definition the logarithm base-n of m as

$$\log_n m = k$$

In english, the logarithm $\log_n m$ asks "n to what power is m?" In this case, that is k, so we write $\log_n m = k$. The subscripted number is called the "base." If not base is written, base-10 is assumed.

Definition (Natural Logarithm). The natural logarithm is a logarithm in taken in base-e. Instead of writing $\log_e x$, one may write $\ln x$.

Example 1.

- $\log_3 9 = 2$ because $3^2 = 9$. Try to come up with five other examples of simple logarithms like this.
- $\log 100 = 2$ because $10^2 = 100$. In this example, the base is omitted, so we assume base-10.
- $\log_8 2 = \frac{1}{3}$ because $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$. If this example isn't clear, go back to exponent properties
- $\ln e = 1$ because $e^1 = e$. Indeed, $\log_n n = 1$ for any n

♢3 Main Lines

Each of the following theorems can be proven very quickly using other properties. In fact, I would encourage you to try them out yourself once you understand the proofs provided. The proofs involving other log properties don't provide much intuition into any exponential origin. For concreteness and clarity, I will show direct proofs from exponent properties, since logarithm properties are practical extensions of these facts.

Theorem: Unofficial Fundamental Property of Logarithms

$$n^{\log_n m} = m$$

Proof. Unofficial Fundamental Property of LogarithmsRecall that $\log_n m$ denotes the power of n that is equal to m. Specifically, if we let $\log_n m = x$, then $n^x = m$ by the very definition of the logarithm. Therefore,
$$n^{\log_n m} = m$$

This is by far the most common logarithm property used in problem solving. Through this result, a bridge is built from logarithms to exponents, providing the main intuition as to what exactly a logarithm means. Study this closely, for the rest of this handout relies on your full understanding of this idea.

Problem (Example). Let $a, b, c, d \in \mathbb{N}$. What is the value of
$$a^{\log_a b+\log_a c+\log_a d}$$
if a, b, c, d are the first four prime numbers respectively?

Solution. We could solve for $\log_2 3$, $\log_2 5$, and $\log_2 7$, but it is way easier to reduce our work as follows:
$$a^{\log_a b+\log_a c+\log_a d} = a^{\log_a b} \cdot a^{\log_a c} \cdot a^{\log_a d} = bcd = \boxed{105}$$
By interpreting the logarithms in their exponential meanings, we can dramatically reduce the amount of work we have to do to solve the problem.

Theorem: Property of Logarithms

Let $a, b, c \in \mathbb{R}$. Then, we have
$$\log_a b + \log_a c = \log_a bc$$

Proof. Let $x = \log_a b$ and $y = \log_a c$. Notice that $a^x = b$ and that $a^y = c$. So,
$$bc = a^x b^y = a^{x+y}$$
Taking the logarithm base-a of both sides, we get
$$\log_a bc = \log_a a^{x+y}$$
Now, what power do we raise a to get a^{x+y}? That's just $x + y$ so we have that
$$\log_a bc = x + y = \log_a b + \log_a c$$
As desired.

Here, we have a great tool to split apart factors inside of a logarithm and bind together same-base logarithms through addition. For instance, $\log_2(2x) = \log_2 2 + \log_2 x = 1 + \log_2 x$. While solving problems, this property is leveraged to reduce equations and force nice cancellations.. Consider the following problem as an example:

Problem (2003 AMC 12B Problem 17). If $\log(xy^3) = 1$ and $\log(x^2y) = 1$, what is $\log(xy)$?

Let's add the two equations to get
$$\log(xy^3) + \log(x^2y) = 2 \implies \log(x^3y^4) = 2$$
We can quickly see that adding $\log(x^2y) = 1$ once more gives us what we are looking for! Namely,
$$\log(x^3y^4) + \log(x^2y) = 3 \implies \log((xy)^5) = 3 \implies 10^3 = (xy)^5$$
Taking the fifth root on both sides reveals that $xy = 10^{\frac{3}{5}}$. After taking logs base-10 on both sides, we see that
$$\log(xy) = \log(10^{\frac{3}{5}}) = \frac{3}{5}.$$
If you return to this problem after reading property #4, you'll notice a quicker exit door to this problem:
$$\log((xy)^5) = 5\log(xy) = 3 \implies \log(xy) = \frac{3}{5}.$$

Theorem: Property of Logarithms

Suppose that $a, b, n \in \mathbb{R}$. Then,

$$\log_a b^n = n \log_a b$$

Proof. Let $x = \log_a b$ and $y = \log_a b^n$. Notice that $a^x = b$ and that $a^y = b^n$. It follows that

$$a^{xn} = a^y.$$

Taking logs base-a on both sides gives

$$xn = y \quad \text{which is equivalent to} \quad n \log_a b = \log_a b^n$$

Indirectly, this property is just restating the fact that raising a number of the form n^m to the power of k -- for instance -- is the same as raising n to the power of mk (for some n, m, k). Consider the following problem asking about exercising this tech:

Problem (New Tech Demo). A poor kitty, denoted k (which is also a member of $\mathbb{R}$ for some reason), is stuck in a tree:

$$tree = t^{t^{t^k}}$$

How can you safely get the kitty to ground level if ``ground level'' is definitiond by the aspect of not being in the power of another number (such as t in this case) or being raised to any power? (This is by no means rigorous, it is a mere exercise of a logarithm property)

Solution. Poor kitty, fear not, the lumberjack is here! I will take logs from this tree (sorry) until you are safe. Allow me to pull out my handy tool, $f(x) = \log_{t^{t^t}} x$. After one quick application of my tool,

$$f(tree) = \log_{t^{t^t}}(tree) = \log_{t^{t^t}} t^{t^{t^k}} = k$$

Now, the kitty is safe on the ground level.

Theorem: Property of Logarithms

Let $a, b, c \in \mathbb{R}$. Then, we have

$$\log_a b - \log_a c = \log_a \left(\frac{b}{c}\right)$$

Proof. Property of LogarithmsAgain, definition $x = \log_a b$ and $y = \log_a c$. Since, $a^x = b$ and $a^y = c$,it follows that

$$\frac{b}{c} = \frac{a^x}{a^y} = a^{x-y}$$

Now, taking logs base-a on both sides yields

$$\log_a \left(\frac{b}{c}\right) = \log_a a^{x-y} = x - y = \log_a b - \log_a c$$

as desired.

This result is very similar to the addition of logarithms property. Instead, you have a method to handle subtraction. This property is very versatile because you can break apart single logarithms into divisions and then split them apart. For instance, it is logical to say $\ln(3) = \ln\left(\frac{27}{9}\right) = \ln(27) - \ln(9)$. Going backwards, it many pose equally helpful or useful to say that $\ln(27) - \ln(9) = \ln\left(\frac{27}{9}\right) = \ln(3)$.

Problem (MAθ 1992). If $\log 36 = a$ and $\log 125 = b$, express $\log(1/12)$ in terms of a and b.

Solution. Notice that

$$\log\frac{1}{12} = \log\frac{1}{2} + \log\frac{1}{6} = (\log 1 - \log 2) + (\log 1 - \log 6) = -\log 2 - \log 6$$

The first logarithm can be rewritten as $\log 2 = \log\frac{10}{5} = \log 10 - \log 5 = 1 - \log 5$. Now, since $a = \log 36$ and $36 = 6^2$, we have that $\log 36 = \log 6^2 = 2\log 6$ so $\frac{a}{2} = \log_6$. Similarly, $\log 125 = \log 5^3 = 3\log 5 = b$, so $\frac{b}{3} = \log 5$. Plugging everything in, we get that

$$\log(1/12) = -(1 - \log 5) - \log 6 = \frac{a}{2} + \frac{b}{3} - 1$$

Solution (Alternate). I can use a different tool: $f(x) = \log_t x$ and get the kitty down with

$$f(f(f(tree))) = f(f(t^{t^k})) = f(t^k) = k$$

In general, this property allows you to extract powers from the argument and get them as constants. Likewise, you can always take a coefficient of a logarithmic term such as $z\log_x y$ and place the coefficient inside the logarithm with $\log_x y^z$.

Theorem: Property of Logarithms

Suppose that $a, b, c \in \mathbb{R}$. Then,

$$(\log_a b)(\log_b c) = \log_a c$$

Proof. Property of LogarithmsLet $x = \log_a c$ and consider the fact that $c = a^x$. Taking logs base-b on both sides gives

$$\log_b c = \log_b a^x$$

Now, let $y = \log_b a$ and $z = \log_b a^x$. Since $b^{yx} = b^z$, we have that

$$yx = z \quad \text{so} \quad (\log_b a)(\log_a c) = \log_b a^x.$$

Since $\log_b a^x = \log_b c$ we have that

$$(\log_a b)(\log_b c) = \log_a c.$$

The chain rule of logarithms is also very related to the change of base property. Specifically, dividing by the term with the same base as the other side derives the change of base property instantly. To see when to use this property, keep an eye on products of logarithms where the base of one logarithm is in the argument of another. For instance, consider the following problem:

Problem (Original). Assume $n \in \mathbb{N}$ such that $n > 2$. Evaluate the product

$$\prod_{i=2}^{n}\prod_{j=2}^{n}\log_i j$$

Solution Notice that for any p, q such that $2 \leqslant p, q \leqslant n$, $\log_p q$ and $\log_q p$ appear exactly once in the product. Since $\log_p q \cdot \log_q p = \log_p p = 1$, it is possible to arrange the product such that each logarithm is multiplied with a logarithm with swapped bases and arguments. Therefore, the product is $\boxed{1}$.

Theorem: Property of Logarithms

Suppose that $a, b, n \in \mathbb{R}$. Then,

$$\log_{a^n} b^n = \log_a b$$

Proof. Property of LogarithmsLet $x = \log_a b$. It is sufficient to show that $a^{xn} = b^n$. Since $a^x = b$, we are done.

In solving problems involving logs, spotting powers in the base and argument can be a sign to leverage this property. An example of a use-case for this property can be seen in the 2021 AMC12A on problem 14.

Problem (AMC12 2021A). What is the value of

$$\left(\sum_{k=1}^{20} \log_{5^k} 3^{k^2}\right) \cdot \left(\sum_{k=1}^{100} \log_{9^k} 25^k\right)?$$

Solution. We can notice the fact that both the base and argument of both logarithms are raised to the power of some power of k. Since $\log_{a^n} b^n = \log_a b$ for $a, b, n \in \mathbb{R}$, we can see that

$$\log_{5^k} 3^{k^2} = \log_{5^k} 3^{k \cdot k} = k \log_{5^k} 3^k = k \log_5 3.$$

Now, we can rewrite the first factor as

$$\log_5 3 \sum_{k=1}^{20} k$$

Recall that $\sum_{i=1}^{n} = \frac{n(n+1)}{2}$, so the first factor reduces further:

$$\log_5 3 \sum_{k=1}^{20} k = \log_5 3 \left(\frac{20(21)}{2}\right) = 210 \log_5 3.$$

The logarithm inside of the second term immediately loses the k in the argument and base by the same property. Therefore,

$$\log_{9^k} 25^k = \log_9 25.$$

We aren't done! Notice that

$$\log_9 25 = \log_{3^2} 5^2 = \log_3 5$$

So, the second term transforms as follows:

$$\sum_{k=1}^{100} \log_{9^k} 25^k = \sum_{k=1}^{100} \log_3 5 = 100 \log_3 5.$$

Putting it all together, we have

$$\left(\sum_{k=1}^{20} \log_{5^k} 3^{k^2}\right) \cdot \left(\sum_{k=1}^{100} \log_{9^k} 25^k\right) = (210 \log_5 3)(100 \log_3 5) = 21000(\log_5 3 \cdot \log_3 5).$$

Finally, since $(\log_a b)(\log_b c) = \log_a c$ for $a, b, c \in \mathbb{R}^+$, it follows that $\log_5 3 \cdot \log_3 5 = \log_5 5 = 1$. Our final answer is thus $21000(\log_5 3 \cdot \log_3 5) = 21000 \cdot 1 = \boxed{21000}$.

Theorem: Property of Logarithms

Suppose that $a, b, c, d \in \mathbb{R}$. Then,

$$(\log_a b)(\log_c d) = (\log_a d)(\log_c b)$$

Proof. Property of LogarithmsWe let $x = \log_a b$, $y = \log_c d, w = \log_a d$, and $z = \log_c b$. Notice that

$$b = a^x = c^z \text{ and } d = a^w = c^y,$$

so $a = c^{(z/x)}$ and $a = c^{(y/w)}$, which gives $c^{(z/x)} = c^{(y/w)}$. Thus we have

$$\frac{z}{x} = \frac{y}{w} \implies xy = wz \implies (\log_a b)(\log_c d) = (\log_a d)(\log_c b)$$

as desired.

This property is a bit difficult to spot in problems. Usually, this rule is used in order to swap arguments in a product of logarithms for the purpose of cancellation. For instance, consider the following problem:

Problem (Original). Simplify the following:

$$\log_e b \cdot \log_a d \cdot \log_d c \cdot \log_b a \cdot \log_c e$$

We can just trade arguments until each base has the same value as the argument. For instance, we can notice that $\log_e b \cdot \log_c e = \log_e e \cdot \log_c b = \log_c b$. Then, we can repeat with $\log_c b \cdot \log_d c$ and so on. Eventually, this all reduces to 1.

Theorem: Property of Logarithms

Suppose that $a, b, c \in \mathbb{R}$. Then,

$$\frac{\log_a b}{\log_a c} = \log_c b$$

Proof. Property of LogarithmsLet $x = \log_a b$, $y = \log_a c$, and $z = \log_c b$. Notice that $a^y = c$. Raising both sides to the power of z gives $a^{yz} = c^z$. But since $c^z = b = a^x$, we have that $a^{yz} = a^x$. Taking logs base-a gives

$$yz = x \quad \text{so} \quad y = \frac{x}{z} \implies \frac{\log_a b}{\log_a c} = \log_c b$$

This powerful result can easily allow you to unify different-base logarithms through one singular base. Considering which base to change everything to is an important step: while one substitution may end up in everything reducing to simpler terms, changing to another base may result in an algebraic nightmare.

Problem (AHSME). If $a > 1$, $b > 1$, and $p = \dfrac{\log_b(\log_b a)}{\log_b a}$, then find a^p in simplest form.

Solution. There's no doubt that p looks a bit ugly in its current state. We would like to hope that there would be a way to reduce p in some way. Indeed, since the numerator and denominator share the same base, we kill two birds with one stone by simplifying p and turning it into a logarithm in base-a to allow for nice cancellation when finding a^p. Namely, we can use the change of base property of logarithms to see that

$$\frac{\log_b(\log_b a)}{\log_b a} = \log_a(\log_b a) \quad \text{so} \quad a^p = a^{\log_a b} = b.$$

Theorem: Property of Logarithms

Suppose that $a, b \in \mathbb{R}$. Then,

$$\log_a b = \frac{1}{\log_b a}$$

Proof. Property of LogarithmsLet $x = \log_a b$ and $y = \log_b a$. Clearly, $a^x = b$ and $b^y = a$ so

$$a^{xy} = (a^x)^y = b^y = a.$$

Taking logs base-a on both ends yields $xy = 1$ so

$$(\log_a b)(\log_b a) = 1 \implies \log_a b = \frac{1}{\log_b a}$$

This property introduces a nice, simple way to switch the base and the argument. In many cases, this property is leveraged in order to extract logarithms from denominators or cancel like terms. Consider the following example:

Problem (2021 AMC 12B P9). What is the value of
$$\frac{\log_2 80}{\log_{40} 2} - \frac{\log_2 160}{\log_{20} 2}?$$

Solution. Notice how convenient it **would** be if all of the logarithms had the same base. It turns out, we are lucky in this scenario because the arguments of the non-base-2 logarithms are both 2, meaning some inverting bases and arguments could help us out. Indeed, notice that
$$\begin{aligned}\frac{\log_2 80}{\log_{40} 2} - \frac{\log_2 160}{\log_{20} 2} = \frac{\log_2 80}{\frac{1}{\log_2 40}} - \frac{\log_2 160}{\frac{1}{\log_2 20}} &= \log_2 80 \cdot \log_2 40 - \log_2 160 \cdot \log_2 20 \\ &= (\log_2 4 + \log_2 20)(\log_2 2 + \log_2 20) - (\log_2 8 + \log_2 20)\log_2 20 \\ &= (2 + \log_2 20)(1 + \log_2 20) - (3 + \log_2 20)\log_2 20\end{aligned}$$
For simplicity sake, let $x = \log_2 20$, then, we have
$$(x+2)(x+1) - x(x+3) = x^2 + 3x + 2 - (x^2 + 3x) = \boxed{2}$$

♢4 Advanced Tactics

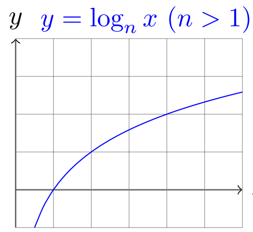
♢4.0.1 What's Fair Game?

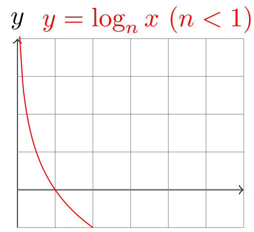
Note that there will never be solutions to $\log_1 k$, even for $k = 1$. A common misconception is to assume that $\log_1 1$ has infinite solutions in $\mathbb{R}$, but this isn't true because
$$\log_1 1 = \frac{\ln 1}{\ln 1} = \frac{0}{0} \quad \text{which is indeterminate.}$$
Likewise, $\log_0 k$ is also undefinitiond, and it is not true that $\log_0 0$ has infinitely many solutions in $\mathbb{R}$ for a similar line of reasoning. definition $f(x) = \log_n x$. Then, f is undefinitiond for $n < 0$, for if it wasn't, logarithms would have to be definitiond in $\mathbb{C}$. There's another name for that, called complex logarithms. Complex logarithms will actually yield solutions for equations like $1^x = 2$. But, for now, we must be real lumberjacks.

If $x > 0$ in $f(x) = \log_n x$, then
$$\log_n -x = \log_n(-1 \cdot x) = \log_n -1 + \log_n x = \frac{\ln -1}{\ln n} + \log_n x = \frac{i\pi}{\ln n} + \log_n x \quad \text{because} \quad e^{\pi i} = -1$$

♢4.0.2 Undoing e^x and $\log_n x$

An alternate -- but rather important -- definition of the logarithm is that is the inverse of the exponential function when the base is nonnegative and not equal to 1. Namely, if $f(x) = n^x$ then $f^{-1}(x) = \log_n x$ for $n > 1$ because
$$f(f^{-1}(x)) = n^{\log_n x} = x \quad \text{and} \quad f^{-1}(f(x)) = \log_n n^x = x.$$





◊4.0.3 Bounding the Logarithm

Observing the graph of $f(x) = \log_n x$ for $n > 1$, a few handy tools arise. Firstly, we have that

$$0 < x < 1 \implies f(x) < 0 \quad \text{and} \quad x \geqslant 1 \implies f(x) \geqslant 0.$$

To prove the former, let $p, q \in \mathbb{R}^+$. WLOG, fix $q > 1 > p > 0$ and let $f(x) = \log_p x$. Consider $f(x) = \log_p x$, then, $p^{f(q)} = q$. Assume $f(x) > 0$ for $0 < x < 1$, then, $p^{f(q)} < 1$. But this is a contradiction because $p^{f(q)} = q$ and $q > 1$, so $f(x) < 0$ for $0 < x < 1$. For the ladder, suppose p, q are non-negative reals with $q > p > 1$. Consider $f(x) = \log_p x$ and the fact that $p^{f(q)} = q$. Assume, $f(x) < 0$ for $x \geqslant 0$, then, we have two cases. Either $f(q) = 0$ or $f(q) > 0$. If $f(q) = 0$, then we have a contradiction because $p^{f(q)} = 1$ and $q > 1$. If $f(q) > 0$, we have another contradiction since $p^{f(q)} < 1$ and $q > 1$, so $f(x) \geqslant 0$ for $x \geqslant 0$.

Now, peering at the graph of $f(x) = \log_n x$ for $n < 1$, we attain some similar results. We have that

$$0 < x < 1 \implies f(x) > 0 \quad \text{and} \quad x \leqslant 1 \implies f(x) \leqslant 0.$$

Indeed, the proofs are very similar to the previous with converse logic; I challenge you to try them yourself.

Another nice result comes from the fact that $\log_n x$ (with $(n > 1)$) is strictly monotonically increasing. Namely, if $p, q \in \mathbb{R}^+$ then

$$p > q \iff \log_n p > \log_n q$$

To prove this, it suffices to show that f is monotonically strictly increasing (which is the same as showing $f'(x)$ is positive for $x > 0$) can see that if $f(x) = \ln x$ then $f'(x) = \frac{1}{x}$ which is positive for all $x > 0$.[1]

The final result involving inequalities involves abusing the fact that logarithms are convex functions for bases greater than 1 and convex for bases less than 1. Namely, if $f(x) = \log_n x$ and $n > 1$, $f''(x) \geqslant 0$ and if $f(x) = \log_n x$ and $n < 1$, $f''(x) \leqslant 0$. Why is this important?

Definition (Jensen's Inequality). Let f be a convex function on the interval $[a, b]$. It follows that

$$f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \leqslant \frac{f(a_1) + f(a_2) + \cdots + f(a_n)}{n}$$

for all $a_i \in [a, b]$. If f is concave, then the inequality flips.

This means that for $n > 1$ in $\log_n x$, the following inequality holds for each $a_i \in [a, b]$:

$$\log_n\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \leqslant \frac{\log_n(a_1) + \log_n(a_2) + \cdots + \log_n(a_n)}{n}$$

Likewise, $n < 1$ in $\log_n x$, the following inequality holds for each $a_i \in [a, b]$:

$$\log_n\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right) \geqslant \frac{\log_n(a_1) + \log_n(a_2) + \cdots + \log_n(a_n)}{n}$$

[1] If you aren't familiar with derivatives, you can disregard the proof for now. Intuitively the proof is leveraging the fact that $\ln(x + h) > \ln(x)$ for $x, h > 0$ to show that the inequality holds for any positive m, n such that $m > n$

Problem (Use-case of Jensen's). Let $a, b, c > 0$. Prove that

$$a^a b^b c^c \geqslant \left(\frac{a+b+c}{3}\right)^{a+b+c}$$

This problem can seem intimidating, especially since it asks you to prove the inequality. But fear not, for this problem is quickly demolished by considering the function $f(x) = x\ln(x)$. Namely,

$$\text{since } f''(x) = \frac{1}{x}, \quad f''(x) \geqslant 0 \quad \forall x \geqslant \quad 0 \text{ so}, \quad f \quad \text{is a convex function.}$$

Now, we can make f work for us leveraging Jensen's inequality. Specifically,

$$\frac{a\ln(a) + b\ln(b) + c\ln(c)}{3} \geqslant \left(\frac{a+b+c}{3}\right)\ln\left(\frac{a+b+c}{3}\right).$$

With just a bit of algebra, we see that this is equivalent to

$$\ln(a^a b^b c^c) \geqslant \ln\left(\left(\frac{a+b+c}{3}\right)^{\frac{a+b+c}{3}}\right) \implies a^a b^b c^c \geqslant \left(\frac{a+b+c}{3}\right)^{a+b+c} \quad \text{after raising } e \text{ to the power of each side.}$$

♢4.0.4 Cutting log Systems into Exponential Equations

A tactic you probably saw during the proofs of the logarithm properties is the ability to interpret logarithms as exponential equations. Usually, putting logarithms in this form can make them more flexible to work with. You need not seek out logarithmic properties for reduction, and instead, you can just use some standard algebra to reach a solution.

Problem (2020 AIME II P3). The value of x that satisfies $\log_{2^x} 3^{20} = \log_{2^{x+3}} 3^{2020}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution. Let $n = \log_{2^x} 3^{20} = \log_{2^{x+3}} 3^{2020}$, then, we can see that

$$(2^x)^n = 3^{20} \quad \text{and} \quad (2^{x+3})^n = 3^{2020}.$$

If we expand the second equation, we arrive at

$$2^{xn} \cdot 8^n = 3^{2020}$$

Since $2^{xn} = 3^{20}$, we can substitute this in to get

$$3^{20} \cdot 8^n = 3^{2020} \implies 8^n = 3^{2000} \implies 8^{\frac{n}{100}} = 3^{20} \implies 2^{\frac{3n}{100}} = 3^{20}$$

Using the first equation, we can see that

$$2^{xn} = 2^{\frac{3n}{100}} \implies x = \frac{3}{100}$$

So the answer is $100 + 3 = \boxed{103.}$

♢4.0.5 Addressing the Tingly Sense

Before we head on, you may not be quite satisfied with the solution to the last problem. Let's take one more look:

Problem (2020 AIME II P3). The value of x that satisfies $\log_{2^x} 3^{20} = \log_{2^{x+3}} 3^{2020}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

You may have thought to yourself ``I wonder if we can work with the fact that $\log_{a^n} b^n = \log ab$...'' This may have been inspired by the fact that both the base and argument are risen to a power. This also could've been inspired by the fact that the base and the argument are the same on the LHS and RHS. If you felt/thought this, good! This is your intuition of logarithms speaking to you and it turns out that this ``tingly sense'' yields a very nice solution.

We first ask what is preventing us from just using the fact that $\log_{a^n} b^n = \log ab$. Indeed, it's because in our problem, the power of the base in the LHS, (x) does not match the power of the argument (20). The same holds for the LHS. Let's explore a slightly adjusted version of the property we are trying to use that matches our scenario more closely. Namely, we may ask what something like $\log_{a^x} b^y$ reduces to (because in an ideal world, these variables would be the values in the original problem). Since x, y are relatively positive integers, there must be some rational number k for which $y = kx$. So, we can rewrite our logarithm as

$$\log_{a^x} b^{kx} = k \log_{a^x} b^x = k \log_a b.$$

Since $k = \frac{y}{x}$, we've now shown that

$$\log_{a^x} b^y = \frac{y}{x} \log_a b.$$

Solution (Quicker). This problem is quickly obliterated by the fact that $\log_{a^x} b^y = \frac{y}{x} \log_a b$. Namely, we transform the original equation into

$$\frac{20}{x} = \frac{2020}{x+3} \implies x = \frac{3}{100}.$$

Then, the problem finishes the same as before.

♢4.0.6 Substituting Everything to Oblivion

There are some cases where it simply doesn't make sense to work with logarithms or exponents in a system of equations. It is in these cases where making a small substitution or two could reduce a TON of algebraic grunt work. It's the difference between writing an essay solution consisting of many carefully selected algebraic manipulations, or reducing your work to a few lines. Consider AIME problem #2 from 2023:

Problem (2023 AIME I P2). Positive real numbers $b \neq 1$ and n satisfy the equations

$$\sqrt{\log_b n} = \log_b \sqrt{n} \qquad \text{and} \qquad b \cdot \log_b n = \log_b(bn).$$

The value of n is $\frac{j}{k}$, where j and k are relatively prime positive integers. Find $j + k$.

Here, we are given a system of equations and asked to effectively solve for n. However, after a recall of the logarithmic properties, it appears nothing will work nicely for us. As you will see, when there are no quick tricks to pull out of the bag, It is far more worth your time to consider some substitutions that could dramatically reduce the amount of work that goes into your solution.

Solution (Take 1). Notice the second equation: $b \cdot \log_b n = \log_b(bn)$. It isn't difficult to see that

$$\log_b(bn) = \log_b n + \log_b b = \log_b n + 1.$$

Now, since

$$b \cdot \log_b n = \log_b n + 1$$

We can divide both sides by $\log_b n$ to arrive at

$$b = \frac{\log_b n}{\log_b n} + \frac{1}{\log_b n} = 1 + \log_n b \implies \log_n b = b - 1$$

Letting the base of both sides be n, our equation reveals that

$$n^{b-1} = b.$$

Notice that since $b = n^{b-1}$, we can see substitute n^{b-1} into the base of the first equation to get an equation in b. Namely,

$$\sqrt{\log_b n} = \log_b \sqrt{n} \quad = \quad \sqrt{\log_{n^{b-1}} n} = \log_{n^{b-1}} n^{\frac{1}{2}}$$

With a bit of algebra, this equation reduces further:

$$\sqrt{\frac{1}{b-1}} = \frac{1}{2b-2} \quad = \quad \frac{1}{b-1} = \frac{1}{4b^2 - 8b + 4} \quad = \quad 4b^2 - 9b + 5 = 0.$$

Now, the solutions to this quadratic are $b = 1, \frac{5}{4}$, and since $b \neq 1$, that must mean $b = \frac{5}{4}$. Now, plugging this into $n^{b-1} = b$ and solving for n gives:

$$n^{\frac{1}{4}} = \frac{5}{4} \implies n = \frac{625}{256}.$$

Since this fraction is in simplest terms, the numerator and denominator are relatively prime, so the answer is $625 + 256 = \boxed{881.}$

Solution (Take 2). Let $x = \log_b n$ and the problem is destroyed. From the original equations, we have

$$\sqrt{x} = \frac{1}{2}x \quad \text{and} \quad bx = 1 + x$$

With a tiny bit of algebra, $x = 4$ and $b = \frac{5}{4}$. So, we have

$$n = b^x = \frac{625}{256}$$

which finishes exactly the same as before.

♢4.0.7 Counting Digits

If $d(x)$ is a function that gives you the number of digits in x, one option is to definition $d(x) = \lfloor \log(x) \rfloor$. To see why this is the case, suppose we have a bunch of integers $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ where each a_i satisfies $0 \leqslant a_i \leqslant 9$ if $i \neq n$ and $0 < a_i \leqslant 9$ if $i = n$. Then,

$$\sum_{i=0}^{n} 10^i a_i \text{ denotes a number where each } a_i \text{ is the } i\text{-th digit.}$$

In this case, you should be able to see that there are n digits, so we would like to show that

$$\left\lfloor \log\left(\sum_{i=0}^{n} 10^i a_i\right) \right\rfloor = n.$$

Notice that

$$\sum_{i=0}^{n} 10^i a_i = 10^n \left(\sum_{i=0}^{n} \left(\frac{1}{10}\right)^i a_i\right) \implies \left\lfloor \log\left(\sum_{i=0}^{n} 10^i a_i\right) \right\rfloor$$

$$= \left\lfloor \log\left(10^n \left(\sum_{i=0}^{n} \left(\frac{1}{10}\right)^i a_i\right)\right) \right\rfloor = \left\lfloor \log 10^n + \log\left(\sum_{i=0}^{n} \left(\frac{1}{10}\right)^i a_i\right) \right\rfloor = \left\lfloor n + \log\left(\sum_{i=0}^{n} \left(\frac{1}{10}\right)^i a_i\right) \right\rfloor.$$

Now, it suffices to show that $\log\left(\sum_{i=0}^{n} \left(\frac{1}{10}\right)^i a_i\right) < 1$. Expanding $\sum_{i=0}^{n} \left(\frac{1}{10}\right)^i a_i$, we get

$$a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_{n-1}}{10^{n-1}} + \frac{a_n}{10^n}.$$

Taking each a_i maximal (meaning each $a_i = 9$) we have

$$9 + 9\left(\frac{1}{10}\right)^1 + 9\left(\frac{1}{10}\right)^2 + \cdots + 9\left(\frac{1}{10}\right)^{n-1} + 9\left(\frac{1}{10}\right)^n$$

Which is a geometric sequence with starting number 9 and common ratio $\frac{1}{10}$ so the closed form of the sum is

$$\frac{a(1-r^n)}{1-r} = \frac{9(1-\frac{1}{10^n})}{1-\frac{1}{10}} = 10 - 10^{1-n}$$

Since $10^{1-n} > 0$ for any n, it follows that $10 - 10^{1-n} < 10$ Taking logs on both sides reveals that

$$\log(10 - 10^{1-n}) < 1 \implies \log\left(\sum_{i=0}^{n} \left(\frac{1}{10}\right)^i a_i\right) < 1$$

as desired.

♢5 Problems

Problem (AMC12). Let S be the set of ordered triples (x, y, z) of real numbers for which
$$\log_{10}(x+y) = z \qquad \text{and} \qquad \log_{10}(x^2+y^2) = z+1.$$
There are real numbers a and b such that for all ordered triples (x, y, z) in S, we have $x^3+y^3 = a\cdot 10^{3z}+b\cdot 10^{2z}$. What is the value of $a+b$?

Problem (AMC12). Let $m > 1$ and $n > 1$ be integers. Suppose that the product of the solutions for x of the equation
$$8\left(\log_n x\right)\left(\log_m x\right) - 7\log_n x - 6\log_m x - 2013 = 0$$
is the smallest possible integer. What is $m+n$?

Problem (AMC12). Let $a \geq b > 1$. What is the largest possible value of $\log_a \dfrac{a}{b} + \log_b \dfrac{b}{a}$?

Problem (AIME). Let x, y, and z all exceed 1 and let w be a positive number such that $\log_x w = 24$, $\log_y w = 40$, and $\log_{xyz} w = 12$. Find $\log_z w$.

Problem. The sum of the base-10 logarithms of the divisors of 10^n is 792. What is n?